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Analysis of time harmonic wave propagation in an elastic layer under heavy fluid loading

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Abstract

This paper concerns assessment of the validity of elementary models of wave propagation in an isotropic elastic layer under heavy fluid loading as well as analysis of coupling effects due to uneven fluid loading. Dispersion curves, which describe propagation of dominantly flexural 'in-phase' waves and dominantly longitudinal 'anti-phase' waves in an elastic layer loaded by an acoustic medium at both sides, are compared with those obtained in the case of one-sided fluid loading. The latter are also compared with dispersion curves predicted by elementary theories of fluid-loaded plates. It is shown that the compressibility of the fluid dramatically extends the validity ranges of elementary theories and this phenomenon is explained.

1. Introduction

Wave propagation in an elastic plate loaded by an acoustic medium on one side is a classical subject studied in many textbooks and research papers (for example, Refs. [1–4] and the literature cited there). In all these references, the waveguide properties of a plate are described in the framework of an elementary Kirchhoff theory. As it is very well known, this theory has a fairly narrow validity range in the case of vibrations in vacuum, which is, roughly speaking, limited by a cut-on frequency of the second flexural propagating wave (however, this wave is accurately described by Timoshenko theory). Therefore, in the case of heavy fluid loading of a plate the detailed analysis of its waveguide properties is usually performed for not too high frequencies (for a classical 'steel-water' pair, up to the coincidence frequency). Predictions for high frequencies (see Ref. [3]) obtained by use of this model might be regarded of theoretical rather than of practical value due to an anticipated breakdown of the approximate Kirchhoff plate model. One of the purposes of this paper is to show that analysis of propagation of a flexural wave in a plate under heavy fluid loading presented in, for example, Refs. [3,4] is valid in a very broad range of frequencies.

A problem formulation that involves a plate theory (Kirchhoff, Timoshenko or sandwich) does not permit one to distinguish between one-sided and two-sided loading, because the plate equation contains as the loading term a pressure jump over its thickness, regardless of the actual distribution of normal forces at the surfaces of a plate. A problem formulation in the framework of the theory of elasticity is capable of taking into

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account this unevenness of the pressure distribution, which results in coupling between 'in-phase' and 'anti-phase' parts of the transverse motion of surfaces. This 'symmetry breaking' effect has recently been studied in Ref. [5]. The 'anti-phase' motions of surfaces are associated with dominantly longitudinal waves in a fluid-loaded layer and the 'in-phase' motions are related to dominantly flexural waves. In Ref. [6], where no fluid loading is considered, they are referred to simply as longitudinal and flexural waves. The second goal of the present paper is to explore how significant this 'symmetry breaking' effect is in several regimes of wave motion.

To accomplish these tasks, a solution of the benchmark problem of wave propagation in an isotropic elastic layer under one-sided heavy fluid loading is obtained in this paper. Although, the applied methodology is capable of describing waves of travelling and evanescent type, only propagating waves are analysed. To gain a better understanding of the role of fluid loading, the 'symmetry-preserving' case of wave propagation in an elastic layer with identical fluids at both sides is considered. For consistency, an elementary model of propagation of a longitudinal 'anti-phase' wave in a fluid-loaded elastic layer is also suggested and its validity is assessed.

The paper is structured as follows: Section 2 contains solution of the problem of one-sided fluid loading in the framework of the theory of elasticity (plane strain). For convenience, solutions for 'in-phase' and 'anti-phase' cases, when a plate is loaded on both sides, is also given. In this section, the problem of wave propagation in an elastic layer under heavy fluid loading is treated as a perturbation problem and asymptotic analysis of dispersion relation is performed. In Section 3, three simple asymptotic formulas, which describe wave propagation in a Kirchhoff plate below coincidence frequency, at the coincidence frequency and above the coincidence frequency are presented along with an asymptotic formula, which describes propagation of a longitudinal wave in a fluid-loaded plate. The results of direct solution of the problem for an elastic layer are compared with their predictions and with results of asymptotic analysis reported in Section 2. The findings of this paper are summarised in Section 4.

2. Propagation of time-harmonic waves in an elastic layer under heavy fluid

The elastic layer has thickness h, density ρ, Young's modulus E, and Poisson's ratio ν. All motions have frequency ω and time dependence $\exp(-i\omega t)$. The dimensionless frequency parameter is defined as $\Omega = \omega h/c$, where $c^2 = E/\rho$. Then the equations of plane strain (for which $\varepsilon_{zz} = \varepsilon_{zx} = \varepsilon_{zy} = 0$), with non-dimensional lengths scaled on h are

$$(1-\nu)\frac{\partial^2 u}{\partial x^2} + \frac{1-2\nu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial^2 v}{\partial x\,\partial y} + \Omega^2(1-2\nu)(1+\nu)u = 0, \tag{1}$$

$$\frac{1}{2}\frac{\partial^2 u}{\partial x\,\partial y} + \frac{1-2\nu}{2}\frac{\partial^2 v}{\partial x^2} + (1-\nu)\frac{\partial^2 v}{\partial y^2} + \Omega^2(1-2\nu)(1+\nu)v = 0, \tag{2}$$

where $u(x,y)$ and $u(x,y)$ are longitudinal and transverse displacements (see Fig. 1).

An acoustic medium occupies the lower half-space ($-\infty < x < \infty, -\infty < y < -\frac{1}{2}$). The acoustic velocity potential and acoustic pressure, scaled with ωh^2 and E, respectively, are denoted φ and p. They satisfy the following equations:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \left(\frac{\omega h}{c_{\mathrm{fl}}}\right)^2 \varphi = 0, \tag{3}$$

$$p = i\frac{\rho_{\mathrm{fl}}}{\rho}\Omega^2 \varphi, \tag{4}$$

where c_{fl} is the sound speed and ρ_{fl} is the fluid density.

Continuity conditions at the interface $y = -\frac{1}{2}$ have the following form, in which stresses σ_y and τ_{xy} are scaled by Young's modulus

$$\sigma_y = -p, \tag{5a}$$

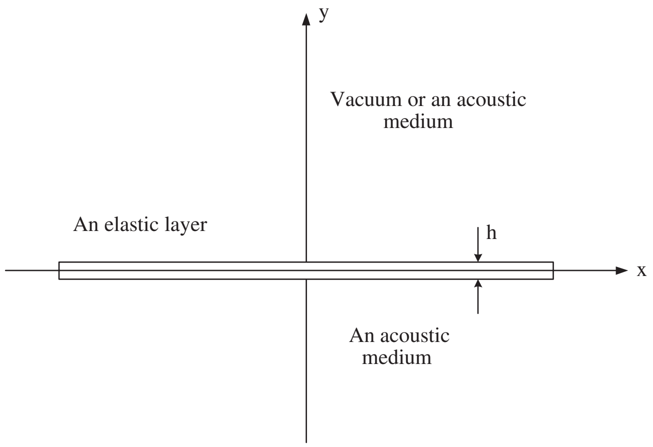

Fig. 1. An elastic layer with one-sided and/or two-sided fluid loading.

$$\tau_{xy} = 0, \tag{5b}$$

$$\frac{\partial \varphi}{\partial y} = -\mathrm{i}v. \tag{5c}$$

Conditions at the upper surface $y = \frac{1}{2}$ (see Fig. 1) are

$$\sigma_y = 0, \tag{6a}$$

$$\tau_{xy} = 0. \tag{6b}$$

The solution of problem (1)–(6) is sought as

$$U(x, y) = [A^{(a)} \cosh(\gamma y) + A^{(\mathrm{in})} \sinh(\gamma y)] \exp(kx), \tag{7a}$$

$$V(x, y) = [B^{(a)} \sinh(\gamma y) + B^{(\mathrm{in})} \cosh(\gamma y)] \exp(kx), \tag{7b}$$

$$\varphi(x, y) = \Phi \exp\left(-\mathrm{i}\sqrt{k^2 + K^2\Omega^2}y\right) \exp(kx), \tag{7c}$$

where $K = c/c_{\mathrm{fl}}$.

The first terms in formulas (7a) and (7b) describe dominantly longitudinal 'anti-phase' motions of surfaces, and the second terms describe their dominantly flexural 'in-phase' motions.

The parameter γ is found from the characteristic equation

$$\left[\frac{1-2v}{2}\gamma^2 + (1-v)k^2 + \Omega^2(1-2v)(1+v)\right]\left[(1-v)\gamma^2 + \frac{1-2v}{2}k^2 + \Omega^2(1-2v)(1+v)\right] - \left(\frac{1}{2}\gamma k\right)^2 = 0. \tag{8}$$

For each root of this equation, γ_n, a modal coefficient $\beta_n \equiv B_n/A_n$ is determined by the relation

$$\beta_n = -\frac{(1-2v/2)\gamma_n^2 + (1-v)k^2 + \Omega^2(1-2v)(1+v)}{(1/2)\gamma_n k} \equiv -\frac{(1/2)\gamma_n k}{(1-v)\gamma_n^2 + (1-2v/2)k^2 + \Omega^2(1-2v)(1+v)}. \tag{9}$$

Then formulas (7a) and (7b) become:

$$\begin{aligned}U(x,y) &= [A_1^{(a)}\cosh(\gamma_1 y) + A_2^{(a)}\cosh(\gamma_2 y)\\ &\quad + A_1^{(\mathrm{in})}\sinh(\gamma_1 y) + A_2^{(\mathrm{in})}\sinh(\gamma_2 y)]\exp(kx),\\ V(x,y) &= [\beta_1 A_1^{(a)}\sinh(\gamma_1 y) + \beta_2 A_2^{(a)}\sinh(\gamma_2 y) + \beta_1 A_1^{(\mathrm{in})}\cosh(\gamma_1 y)\\ &\quad + \beta_2 A_2^{(\mathrm{in})}\cosh(\gamma_2 y)]\exp(kx).\end{aligned}$$

The pressure at the interface is

$$p\left(x,-\frac{1}{2}\right) = \mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2+K^2\Omega^2}}V\left(x,-\frac{1}{2}\right), \tag{10}$$

where $\bar{\rho} \equiv \rho_{\mathrm{fl}}/\rho$.

The dispersion equation emerges from the four conditions (5a), (5b), (6a) and (6b), composing a set of four homogeneous linear algebraic equations, by equating the determinant to zero. It is expedient to present the equations as

$$\begin{aligned}&\left(a_{11}-\tfrac{1}{2}\bar{\rho}b_1\right)A_1^{(a)} + \left(a_{12}-\tfrac{1}{2}\bar{\rho}b_2\right)A_2^{(a)} + \tfrac{1}{2}\bar{\rho}b_3 A_1^{(\mathrm{in})} + \tfrac{1}{2}\bar{\rho}b_4 A_2^{(\mathrm{in})} = 0,\\ &a_{21}A_1^{(a)} + a_{22}A_2^{(a)} = 0,\\ &\tfrac{1}{2}\bar{\rho}b_3 A_1^{(a)} + \tfrac{1}{2}\bar{\rho}b_4 A_2^{(a)} + \left(a_{33}-\tfrac{1}{2}\bar{\rho}b_1\right)A_1^{(\mathrm{in})} + \left(a_{34}-\tfrac{1}{2}\bar{\rho}b_2\right)A_2^{(\mathrm{in})} = 0,\\ &a_{43}A_1^{(\mathrm{in})} + a_{44}A_2^{(\mathrm{in})} = 0,\end{aligned} \tag{11}$$

where

$$a_{11} = (\beta_1\gamma_1 + \nu k)\cosh\frac{\gamma_1}{2}, \quad a_{12} = (\beta_2\gamma_2 + \nu k)\cosh\frac{\gamma_2}{2},$$

$$a_{21} = (k\beta_1 + \gamma_1)\sinh\frac{\gamma_1}{2}, \quad a_{22} = (k\beta_2 + \gamma_2)\sinh\frac{\gamma_2}{2},$$

$$a_{33} = (\beta_1\gamma_1 + \nu k)\sinh\frac{\gamma_1}{2}, \quad a_{34} = (\beta_2\gamma_2 + \nu k)\sinh\frac{\gamma_2}{2}$$

$$a_{43} = (k\beta_1 + \gamma_1)\cosh\frac{\gamma_1}{2}, \quad a_{44} = (k\beta_2 + \gamma_2)\cosh\frac{\gamma_2}{2},$$

$$b_1 = \frac{\mathrm{i}\Omega^2}{\sqrt{k^2+K^2\Omega^2}}\beta_1\sinh\frac{\gamma_1}{2}, \quad b_2 = \frac{\mathrm{i}\Omega^2}{\sqrt{k^2+K^2\Omega^2}}\beta_2\sinh\frac{\gamma_2}{2},$$

$$b_3 = \frac{\mathrm{i}\Omega^2}{\sqrt{k^2+K^2\Omega^2}}\beta_1\cosh\frac{\gamma_1}{2}, \quad b_4 = \frac{\mathrm{i}\Omega^2}{\sqrt{k^2+K^2\Omega^2}}\beta_2\cosh\frac{\gamma_2}{2}.$$

In the absence of fluid loading, $\bar{\rho} = 0$, this system of equations may be factorised into two classical dispersion equations (Rayleigh–Lamb equations), which define propagation of flexural 'in-phase' and longitudinal 'anti-phase' waves in an elastic layer in vacuum (see, for example, Refs. [6,7]). They are presented here for consistency.

The dispersion equation for 'anti-phase' dominantly longitudinal waves is

$$\begin{aligned}D_{10} = a_{11}a_{22} - a_{12}a_{21} &= (\beta_1\gamma_1 + \nu k)(k\beta_2 + \gamma_2)\cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right)\\ &\quad - (\beta_2\gamma_2 + \nu k)(k\beta_1 + \gamma_1)\cosh\left(\frac{\gamma_2}{2}\right)\sinh\left(\frac{\gamma_1}{2}\right) = 0\end{aligned} \tag{12}$$

and the dispersion equation for 'in-phase' dominantly flexural waves is

$$D_{20} = a_{33}a_{44} - a_{34}a_{43} = (\beta_1\gamma_1 + \nu k)(k\beta_2 + \gamma_2)\sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right) - (\beta_2\gamma_2 + \nu k)(k\beta_1 + \gamma_1)\sinh\left(\frac{\gamma_2}{2}\right)\cosh\left(\frac{\gamma_1}{2}\right) = 0. \tag{13}$$

For symmetric fluid loading, i.e., when the lower half-space ($-\infty < x < \infty$, $-\infty < y < -\frac{1}{2}$) and the upper half-space ($-\infty < x < \infty$, $\frac{1}{2} < y < \infty$) are occupied by the same acoustic medium, the dispersion equation factorises as follows for 'anti-phase' waves

$$D_1 = \left[(\beta_1\gamma_1 + \nu k)\cosh\left(\frac{\gamma_1}{2}\right) - \mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2 + K^2\Omega^2}}\beta_1 \sinh\left(\frac{\gamma_1}{2}\right)\right](k\beta_2 + \gamma_2)\sinh\left(\frac{\gamma_2}{2}\right) - \left[(\beta_2\gamma_2 + \nu k)\cosh\left(\frac{\gamma_2}{2}\right) - \mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2 + K^2\Omega^2}}\beta_2 \sinh\left(\frac{\gamma_2}{2}\right)\right](k\beta_1 + \gamma_1)\sinh\left(\frac{\gamma_1}{2}\right) = 0 \tag{14}$$

and for 'in-phase' waves

$$D_2 = \left[(\beta_1\gamma_1 + \nu k)\sinh\left(\frac{\gamma_1}{2}\right) - \mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2 + K^2\Omega^2}}\beta_1 \cosh\left(\frac{\gamma_1}{2}\right)\right](k\beta_2 + \gamma_2)\cosh\left(\frac{\gamma_2}{2}\right) - \left[(\beta_2\gamma_2 + \nu k)\sinh\left(\frac{\gamma_2}{2}\right) - \mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2 + K^2\Omega^2}}\beta_2 \cosh\left(\frac{\gamma_2}{2}\right)\right](k\beta_1 + \gamma_1)\cosh\left(\frac{\gamma_1}{2}\right) = 0. \tag{15}$$

These equations may be written

$$D_1 = D_{10} - \bar{\rho}(b_1a_{22} - b_2a_{21}) = 0, \tag{16a}$$

$$D_2 = D_{20} - \bar{\rho}(b_3a_{44} - b_4a_{43}) = 0. \tag{16b}$$

The dispersion equation for one-sided fluid loading may be expanded in power series in the density ratio $\bar{\rho}$. If only the first two terms are retained, this equation acquires the form

$$D_{10}D_{20} - \tfrac{1}{2}\bar{\rho}[D_{10}(b_3a_{44} - b_4a_{43}) + D_{20}(b_1a_{22} - b_2a_{21})] = 0. \tag{17}$$

This equation to order $\bar{\rho}$ is a product of Eqs. 16(a) and (b) if the two-sided loading is produced by a fluid of the density $(\frac{1}{2})\rho_{\mathrm{fl}}$.

If the density ratio is regarded as a small parameter, then the standard perturbation technique gives the following two-term expansion in Taylor series on $\bar{\rho}$ for a wavenumber:

$$k = k_0 + \left.\frac{\partial k}{\partial \bar{\rho}}\right|_{\bar{\rho}=0}\bar{\rho}. \tag{18}$$

It is convenient to designate roots of dispersion equation for 'anti-phase' and 'in-phase' motions $D_{j0} = 0$ as k_{j0}, $j = 1,2$. Then the 'sensitivity' or the first-order correction $\partial k_j/\partial\bar{\rho}|_{\bar{\rho}=0}$ is defined by

$$\left.\frac{\partial k_1}{\partial \bar{\rho}}\right|_{\bar{\rho}=0} = \frac{1}{2}\frac{(b_1a_{22} - b_2a_{21})}{\partial D_{10}/\partial k}, \tag{19a}$$

$$\left.\frac{\partial k_2}{\partial \bar{\rho}}\right|_{\bar{\rho}=0} = \frac{1}{2}\frac{(b_3a_{44} - b_4a_{43})}{\partial D_{20}/\partial k}. \tag{19b}$$

These formulas suggest that the first-order correction to a wavenumber of the 'in-phase' or 'anti-phase' wave in a plate with one-sided fluid loading is controlled by the same component and is exactly half as small as the correction for two-sided loading (see Eqs. (16a),(16b)).

Therefore the interaction of 'in-phase' and 'anti-phase' waves due to uneven fluid loading manifests itself in terms quadratic in $\bar{\rho}$. However, the analysis presented in this section is restricted by the assumption that

$|\sqrt{k^2+K^2\Omega^2}|$ is sufficiently large. This condition is valid only for relatively low frequencies, i.e., frequencies below coincidence. The structure of the asymptotic expansion of wavenumbers above coincidence is more complicated and discussion of this aspect is presented in Section 3.

3. Simple formulas for wavenumbers in an elastic layer under one-sided heavy fluid loading: comparison with exact results

As is well known, the classical model of Kirchhoff plate approximately describes 'in-phase' dominantly flexural waves in an isotropic elastic layer and its predictions may readily be compared with the results obtained by use of model formulated in the previous section. This is done in the first sub-section. The conventional elementary theory of propagation of plane longitudinal wave in an elastic rod does not allow one to take into account fluid–structure interaction (if viscosity is neglected) because deformation of cross-section is neglected. Therefore, a 'refined' approximate model should be formulated to describe propagation of 'anti-phase' dominantly longitudinal waves, as in the second sub-section below.

Undoubtedly, simplified dispersion equations for various regimes of wave propagation in an elastic layer under heavy fluid loading can conveniently be obtained by means of standard asymptotic techniques. However, it is also possible to derive equations of motion under some heuristic assumptions, which are not necessarily asymptotically consistent with an exact solution. A classical example of such an approach is the formulation of Timoshenko beam theory. In this part of the paper, classical Kirchhoff's model of a plate under heavy fluid loading and 'refined' model of propagation of longitudinal waves in a plate under heavy fluid loading are treated as heuristic theories, that are verified by comparison of their predictions with results obtained in the framework of exact theory presented in Section 2. The rigorous asymptotic analysis of their applicability ranges and their tolerance levels is deliberately left out of the scope of this paper.

3.1. Elementary model of propagation of the 'in-phase' wave

The wavenumber of a propagating 'in-phase' wave in a Kirchhoff plate is formulated as a purely imaginary root of the dispersion equation (it can be found in Refs. [1–4] with different notations)

$$k^4 - 12(1-\nu^2)\Omega^2 - \frac{12(1-\nu^2)\mathrm{i}\bar{\rho}\Omega^2}{\sqrt{k^2+K^2\Omega^2}} = 0. \tag{20}$$

In this paper, the asymptotic expansion with the density ratio $\bar{\rho}$ regarded as a small parameter is explored. Three regimes are identified depending upon the excitation frequency, see Refs. [1–4]. Discussion of scaling applied in these regimes can be found in Section 3.1 of Ref. [4]. For brevity, it is not reproduced here.

Below coincidence, i.e., for $\Omega < \Omega_C \equiv \sqrt{12(1-\nu^2)}/K^2$, the asymptotic formula is

$$k = \mathrm{i}\sqrt{\Omega\sqrt{12(1-\nu^2)}} + \mathrm{i}\frac{\bar{\rho}}{4}\left(1 - \frac{\Omega K^2}{\sqrt{12(1-\nu^2)}}\right)^{-(1/2)} + o(\bar{\rho}). \tag{21}$$

In the coincidence range, the frequency and wavenumber are

$$k = \mathrm{i}\frac{\sqrt{12(1-\nu^2)}}{K}\left[1 + \hat{k}\left(\bar{\rho}\frac{K}{\sqrt{12(1-\nu^2)}}\right)^{2/3}\right] + o(\bar{\rho}^{2/3}), \tag{22a}$$

$$\Omega = \frac{\sqrt{12(1-\nu^2)}}{K^2}\left[1 + \hat{\omega}\left(\bar{\rho}\frac{K}{\sqrt{12(1-\nu^2)}}\right)^{2/3}\right] + o(\bar{\rho}^{2/3}). \tag{22b}$$

The pair of positive numbers $(\hat{k}, \hat{\omega})$ is the root of cubic equation (its derivation is an elementary exercise in *Mathematica* [8] as soon as a dominant balance given in Ref. [4], p. 729 is used):

$$32\hat{k}^3 - 64\hat{k}^2\hat{\omega} + 40\hat{k}\hat{\omega}^2 - 8\hat{\omega}^3 - 1 = 0.$$

At high frequencies, i.e. for $\Omega > \Omega_C \equiv \sqrt{12(1-\nu^2)}/K^2$, the two-term asymptotic formula is

$$k = \mathrm{i}K\Omega\left[1 + \frac{1}{2}\bar{\rho}^2 \frac{\sqrt{12(1-\nu^2)}}{\Omega^2 K^3}\left(\frac{\Omega^2 K^4}{12(1-\nu^2)} - 1\right)^{-2}\right] + o(\bar{\rho}^2). \tag{23}$$

The formulas (21)–(23) approximate the magnitude of the wavenumber of the 'in-phase' dominantly flexural propagating wave in an isotropic elastic layer with one-sided fluid loading determined as Eq. (11).

3.2. Elementary model of propagation of the 'anti-phase' wave

In the framework of an elementary one-dimensional theory of the axial deformation of a straight rod (a plate in the plane problem formulation), deformation of the cross-section is ignored and, therefore there is no influence of the acoustic medium on propagation of an axial wave. An interaction is produced due to the Poisson effect and is captured in the exact formulation of the problem for an elastic layer under heavy fluid loading presented in Section 2. This issue with respect to dynamics of thin-walled elastic bodies has been studied by means of classical asymptotic analysis, for example, in Ref. [9]. However, it is possible to introduce a simple model of propagation of longitudinal wave as for a sandwich plate in Ref. [10]. The predictions of this approximate theory may then be compared with the exact solution.

This model is aimed to describe 'anti-phase' ('breathing') motion of surfaces of an elastic layer and it is convenient to derive this motion for the case of symmetric fluid loading. The approximate theory of propagation of a longitudinal wave in a plate (in plane strain) with fluid loading is based on three assumptions. The first assumption concerns the normal stresses across the thickness of a plate. They are considered as uniformly distributed in y-direction and they balance a pressure acting at the fluid-loaded surfaces, $\sigma_y(x, \pm(\frac{1}{2})) = -p(x, \pm(\frac{1}{2}))$. The second assumption introduces the averaged uniform strain across the thickness, $\varepsilon_{yy}(x) = v(x, \frac{1}{2}) - v(x, -\frac{1}{2})$. The third 'ad hoc' assumption postulates that across-thickness shear stress may be neglected, $\tau_{xy} = 0$. These heuristic simplifications after some algebra (see Appendix A for details) result in the following dispersion equation:

$$k^2\left[\frac{2}{1+\nu} - \mathrm{i}\bar{\rho}\Omega^2\frac{(1-\nu)}{\sqrt{k^2+K^2\Omega^2}}\right]\left[2(1-\nu) - \mathrm{i}\bar{\rho}\Omega^2\frac{(1+\nu)(1-2\nu)}{\sqrt{k^2+K^2\Omega^2}}\right]^{-1} + \Omega^2 = 0. \tag{24}$$

If there is no fluid loading, i.e. if $\bar{\rho} = 0$, this equation yields the conventional elementary formula for a wavenumber of the longitudinal wave in plane strain conditions

$$k_0 = \mathrm{i}\Omega\sqrt{1-\nu^2}. \tag{25}$$

However, propagation of this wave in a fluid-loaded plate may be suppressed due to fluid's compressibility. This effect is readily detected if a perturbation technique is applied and a regular expansion of the root of Eq. (24) in power series on fluid loading parameter is used:

$$k = k_0 + \Lambda_1\bar{\rho} = \mathrm{i}\Omega\sqrt{1-\nu^2} + \Lambda_1\bar{\rho}. \tag{26}$$

Simple algebra gives

$$\Lambda_1 = -\frac{\Omega^2}{4}\frac{1}{\sqrt{K^2-1+\nu^2}}\frac{\nu^2(1+\nu)^2}{\sqrt{1-\nu^2}}. \tag{27}$$

As is seen from this formula, propagation of this 'structure-originated' wave is suppressed as soon as the following condition holds:

$$K \equiv \frac{c}{c_{\mathrm{fl}}} > \sqrt{1-\nu^2}. \tag{28}$$

Naturally, this wave propagates along the plate, if the model of an incompressible fluid is adopted, $K = 0$.

The wavenumber of a propagating wave in a plate loaded by a compressible fluid is defined by another asymptotic formula. Eq. (24) always has the following purely imaginary root:

$$k = \mathrm{i}K\Omega\left[1 - \frac{1}{4}\bar{\rho}^2\frac{\Omega^2}{K^2}\frac{K^2(1-\nu^2)-(1+\nu)^2(1-2\nu)}{1-\nu^2-K^2}\right] + o(\bar{\rho}^2). \tag{29}$$

This formula is obtained by use of the same scaling as formula (23), specifically, $k^2 + K^2\Omega^2 \sim \bar{\rho}^2$, but it is valid in the whole frequency range because the longitudinal wave in a plate is non-dispersive. The wave defined by formula (29) is fluid-originated and decays into the fluid's volume. The wavenumbers given by asymptotic formulas (26) and (29) are compared with numerical solution of dispersion Eq. (14) in the subsequent part of the paper.

The assumptions formulated to derive dispersion relation for the longitudinal 'in-phase' wave in a plate exposed to symmetrical fluid loading become doubtful for one-sided fluid loading. However, formula (29) still gives an accurate result, because the propagating wave is fluid-originated and the residual term is very small indeed. Moreover, there is virtually no difference between results obtained by the use of formulas (29) and (23) in the high frequency range. This means that the wave trapped at the surface of a plate is the same regardless of whether the wave is linked to 'in-phase' or 'anti-phase' motion of the surfaces of an elastic layer, and irrespective of whether one-sided or two-sided fluid loading is considered.

3.3. Comparison of dispersion curves

Consider as the first example wave propagation in a steel plate loaded by water. The density ratio is $\bar{\rho} = 0.128$ and the sound speed ratio is $K = 3.25$. The coincidence frequency parameter is $\Omega_C \approx 0.313$. In Fig. 2a, the dispersion curve for a propagating flexural 'in-phase' wave is presented for one-sided fluid loading. The dispersion curve plotted after three asymptotic formulas (21)–(23) is shown by the thin continuous curve. Its counterpart obtained by numerical solution of the dispersion equation, which follows from the system of Eq. (11), is shown by dots. These curves match each other very well in the whole frequency range. The largest discrepancy between the curves is observed around the coincidence frequency, as seen in the zoomed graph in Fig. 2b. In this figure, curves 1, 2, 3 plotted after (21)–(23), correspondingly, are shown with some overlapping and it is seen that the less accurate approximation is given by the formulas (22a) and (22b) for the coincidence frequency range. Naturally, to recover accuracy, the subsequent terms in the asymptotic expansion on parameter $\bar{\rho}$ should be retained.

This effect is readily explained by inspection into the decay rate $\delta = |\sqrt{k^2 + K^2\Omega^2}|$ of this wave in the acoustic medium (see Fig. 3). Curves 1–3 display this quantity calculated for the pairs (Ω, k) defined by formulas (21)–(23), correspondingly. Dotted curve presents exact solution. As is seen from comparison of curves shown in Figs. 2b and 3, this curve reaches its maximum in the coincidence frequency range, where a discrepancy between exact and asymptotic solutions is larger, than elsewhere. If the decay rate is high, then a volume of acoustic medium involved in wave propagation is minimal. It 'highlights' the difference in behaviour of a fluid-loaded elastic layer and a fluid-loaded Kirchhoff plate that otherwise is smeared out due to a very substantial participation of fluid in this trapped wave motion.

As is seen from these figures, the wave motion above the coincidence frequency is totally controlled by the acoustic medium. The decay rate of a trapped wave in this regime to leading order is estimated in the framework of Kirchhoff theory as

$$\delta = \bar{\rho}\left(\frac{\Omega^2 K^4}{12(1-\nu^2)} - 1\right)^{-1} + o(\bar{\rho}). \tag{30}$$

A difference in the magnitude of the decay rate δ found from the exact solution and that predicted by the asymptotic formula (30) grows with the frequency (see Fig. 3). However, the magnitude of this decay rate is very small in comparison with the magnitude of the wavenumber.

Naturally, the simple Taylor's expansion (18) becomes invalid at high frequencies. Its predictions below the coincidence are compared with the exact solution (dotted curve) in Fig. 4. Dispersion curve after (18)–(19b) is plotted as a thin solid line 2. For convenience, the dispersion curve 1 obtained from formula (21) is also

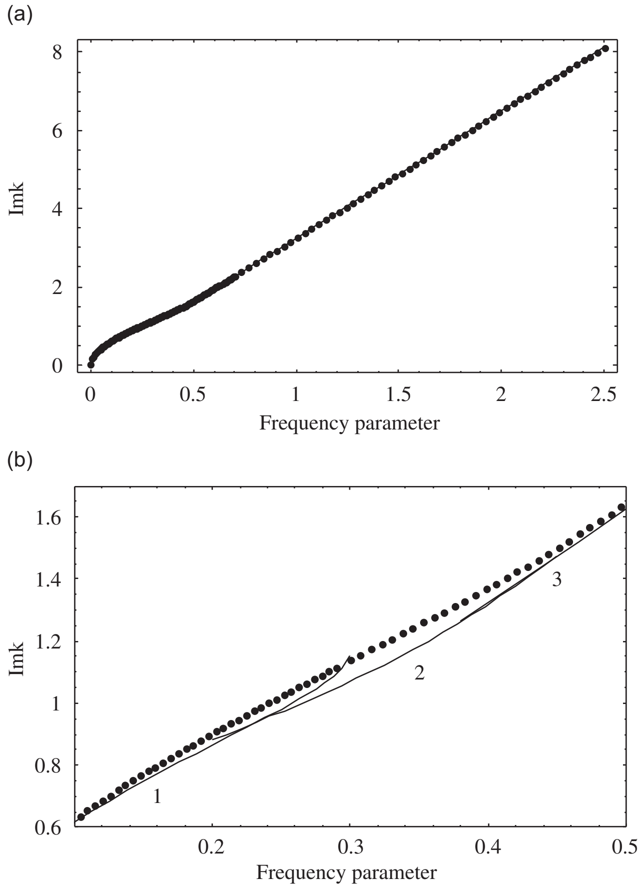

Fig. 2. (a) Dispersion curves Im k for an 'in-phase' wave in an elastic layer with one-sided fluid loading. (b) Zooming of dispersion curves in vicinity of coincidence frequency.

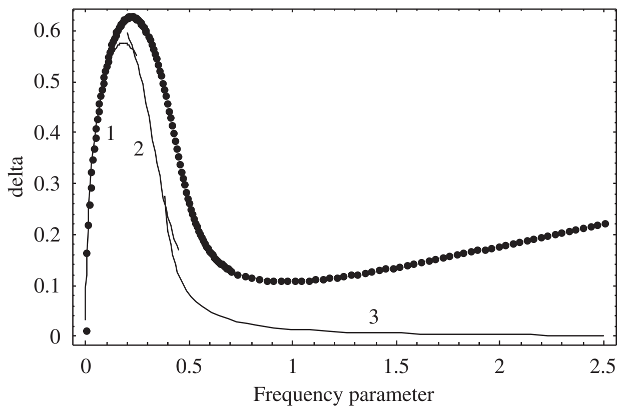

Fig. 3. Dispersion curves δ (decay rate into acoustic volume) for an 'in-phase' wave.

plotted. As is seen, there is almost no difference between the predictions given by formulas (18)–(19b) and (21). They are equally accurate at low frequencies and they fail simultaneously as the excitation frequency approaches the coincidence frequency. If fluid loading is absent, then Kirchhoff plate model becomes invalid at $\Omega \approx 0.25$ (see Fig. 5). In this figure, two dotted curves are plotted after exact Rayleigh–Lamb solution and

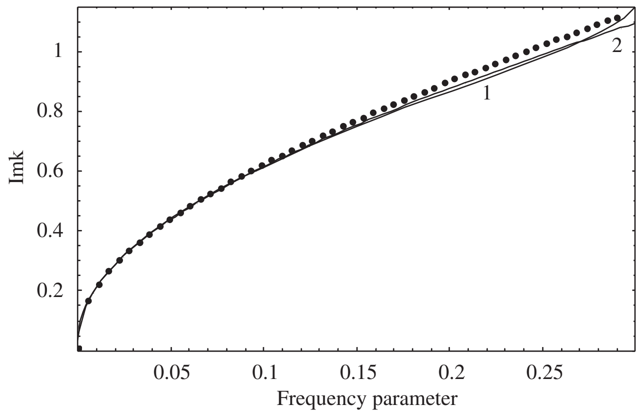

Fig. 4. Dispersion curve Im k for an 'in-phase' wave below coincidence predicted by three theories.

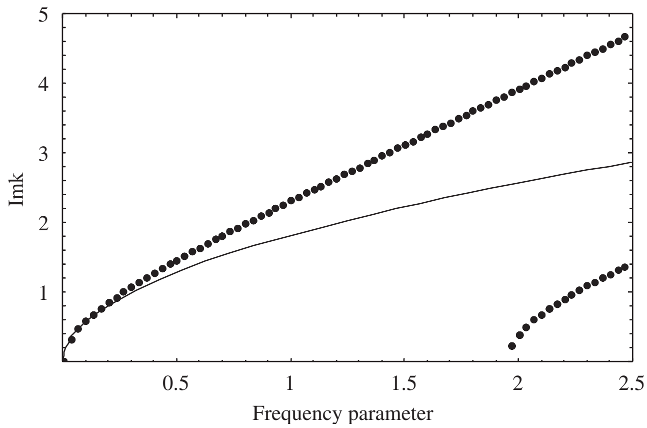

Fig. 5. Dispersion curves Im k for an 'in-phase' wave in an elastic layer without fluid loading.

the thin continuous curve is plotted after Kirchhoff theory. Obviously, this is a trivial result, but this figure is presented here to emphasise the 'accuracy recovering' effect of heavy fluid loading on Kirchoff plate model.

Dispersion curves for the longitudinal 'anti-phase' wave predicted by the elementary formula (26) are shown in Fig. 6a,b as dotted lines. Curve predicted by the elementary formula (25) in the absence of fluid loading is plotted in Fig. 6a as a continuous line. As discussed, propagation of this 'structure-originated' wave is suppressed due to the fluid's compressibility and the real part of wavenumber is presented in Fig. 6b. The wavenumber of a propagating 'fluid-originated' wave as a function of the frequency parameter is presented in Fig. 7a. The prediction of formula (29) and the solution of dispersion Eq. (14) for two-sided fluid loading perfectly match each over in the whole considered frequency range. The same holds true for the decay rate of this trapped wave, see Fig. 7b. In the case of one-sided fluid loading, the formula (19a) becomes less accurate. However, the formula matches the exact solution if the density is reduced to a half of its value. In both cases the fluid-induced correction is very weak for the chosen set of parameters. It may be rather different if $K \sim \sqrt{1-v^2}$, but analysis of this special case lies beyond the scope of the present paper. The dispersion curves in Figs. 2a and 7a virtually merge into each other above the coincidence frequency. Hence at relatively high frequencies, wave propagation in an elastic layer loaded by an unbounded volume of a compressible fluid is controlled by the fluid's properties. To leading order, this holds whether a longitudinal 'anti-phase' or a flexural 'in-phase' wave in a layer is considered, and whether fluid is located on one side or on both sides of the layer.

To assess the practical relevance of the above analysis, wave propagation in an orthotropic laminated plate made of carbon/epoxy UD prepreg in the maximum stiffness direction (along fibres) under one-sided loading

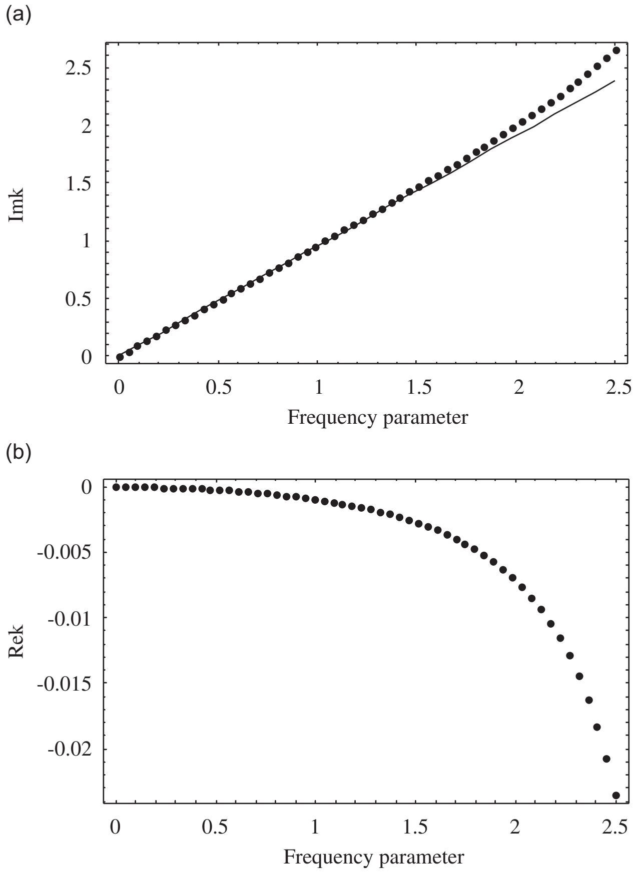

Fig. 6. (a) Dispersion curves Im k for an 'anti-phase' 'structure-originated' wave in an elastic with two-sided fluid loading. (b) Dispersion curves Re k for an 'anti-phase' 'structure-originated' wave in an elastic layer with two-sided fluid loading.

by water is considered. The thickness of the plate is $h = 1\,\text{cm}$. The density of its material is $\rho = 2600\,\text{kg/m}^3$, the Young's modulus along the fibres is $E_{11} = 126\,\text{GPa}$ and the Poisson ratio is $v = 0.28$. Then the non-dimensional fluid loading parameters used in this paper are: $\bar{\rho} = 0.385$, $K = 4.834$. The coincidence frequency is $f_C \approx 15.77\,\text{kHz}$. The dispersion curves in Fig. 8 are plotted in dimensional form. The curves present the dependence of dimensional wavenumber k_{dim} (m^{-1}) on frequency in Hz. As is seen, the predictions from the Kirchhoff theory are in a very good agreement with the exact solution for an elastic layer in the whole frequency as regards the wavenumber (the dotted line virtually overlaps the thin solid line). However, the discrepancy in the estimate of the decay rate is relatively large in the high frequency range, i.e. above coincidence, as has already been shown in Fig. 3. Similarly to the 'steel-water' case, elementary theory underestimates the decay of the wave trapped at the surface of a carbon/epoxy plate. The density ratio in this case is three times larger than in the previous 'steel-water' case. The accuracy of modelling in the framework of the Kirchhoff theory may be improved by taking into account high-order terms in the asymptotic expansions (21)–(23), but this task lies beyond the scope of the present paper.

4. Conclusions

The results reported in this paper may be summarised as follows:

- The validity range of Kirchhoff plate theory for describing propagating dominantly flexural 'in-phase' waves is substantially extended by the heavy fluid loading effects of an unbounded

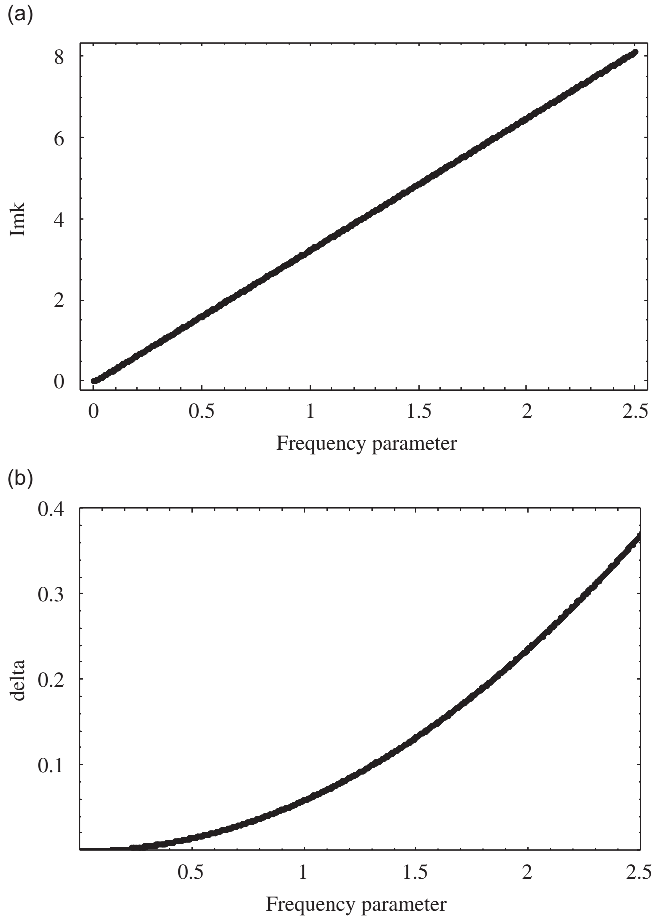

Fig. 7. (a) Dispersion curves Im k for an 'anti-phase' 'fluid-originated' wave in an elastic layer with two-sided fluid loading. (b) Dispersion curves δ (decay rate into acoustic volume) for an 'anti-phase' 'fluid-originated' wave in an elastic layer with two-sided fluid loading.

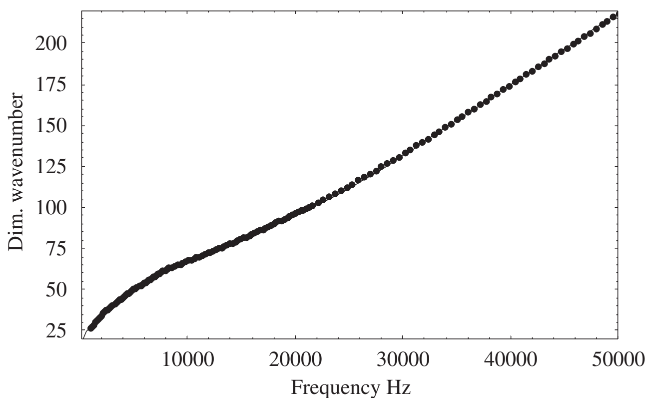

Fig. 8. Dispersion curve for an 'in-phase' wave in an elastic layer with one-sided fluid loading in dimensional form.

volume of an acoustic medium, in comparison with the validity range for time-harmonic behaviour of a plate in vacuum.

- The propagating dominantly longitudinal 'anti-phase' wave in a fluid-loaded elastic layer may be accurately described by an elementary heuristic model.

- Below the coincidence frequency, the coupling between 'in-phase' and 'anti-phase' waves due to uneven (one-sided) fluid loading manifests itself as a second-order correction to the expansion of wavenumber in powers of the density ratio parameter.

As a final remark the analysis presented in this paper concerns free wave propagation in an elastic layer (a plate) under heavy fluid loading. The validity of Kirchhoff plate theory for solving problems involving the forced response of a plate requires further assessment.

Appendix A. Derivation of an approximate dispersion equation for in-phase wave in an elastic layer under both-sides heavy fluid loading

In the case of two-sided fluid loading, the compressive normal stress at the surfaces of a layer is formulated as (see formula (5a) and scaling in Section 2)

$$\sigma_y\left(x, \pm\tfrac{1}{2}\right) = -p\left(x, \pm\tfrac{1}{2}\right). \tag{A.1}$$

This stress is dependent on transverse coordinate y, see Fig. 1, but it is reasonable to assume that its distribution is uniform across thickness

$$\sigma_y(x, y) \approx \sigma_y(x) = -p\left(x, \pm\tfrac{1}{2}\right) \equiv -p_s(x). \tag{A.2}$$

Then Hooke's law specialised for the plane strain ($\varepsilon_{zz} = \varepsilon_{zx} = \varepsilon_{zy} = 0$) case, yields

$$\sigma_z(x) = v\sigma_x(x) - vp_s(x). \tag{A.3}$$

Consequently, the transverse strain appears to be uniform across thickness

$$\varepsilon_{yy}(x) = -(1 - v^2)p_s(x) - v(1 + v)\sigma_x(x). \tag{A.4}$$

On the other hand, in as much as the transverse normal stress is considered as independent upon coordinate y, it is consistent to introduce an averaged transverse strain as

$$\varepsilon_{yy}(x) = v\left(x, \tfrac{1}{2}\right) - v\left(x, -\tfrac{1}{2}\right). \tag{A.5}$$

Due to the symmetry of deformation with respect to mid-surface, it is possible to introduce a function $v_0(x)$ as follows:

$$v\left(x, -\tfrac{1}{2}\right) = -v\left(x, \tfrac{1}{2}\right) \equiv -v_0(x). \tag{A.6}$$

Thus, $\varepsilon_{yy}(x) = 2v_0(x)$ and the axial normal stress is formulated as

$$\sigma_x(x) = \frac{2}{v(1 + v)}v_0(x) - \frac{1 - v}{v}p_s(x), \tag{A.7}$$

Hooke's law

$$\varepsilon_{xx}(x) = (1 - v^2)\sigma_x(x) + v(1 + v)p_s(x) \tag{A.8}$$

yields

$$\varepsilon_{xx}(x) \equiv \frac{\mathrm{d}u(x)}{\mathrm{d}x} = \frac{1 - v}{v}2v_0(x) + \frac{2v^2 + v - 1}{v}p_s(x). \tag{A.9}$$

The differential equation of motion of an infinitely small element of a plate in the longitudinal direction reads as

$$\frac{\mathrm{d}\sigma_x(x)}{\mathrm{d}x} + \rho\omega^2 u(x) = 0. \tag{A.10}$$

It is obtained from the exact equation for a plane strain

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \rho\omega^2 u(x) = 0, \tag{A.11}$$

if shear stresses τ_{xy} are neglected. This is an additional approximation, and its validity should be assessed.

Continuity of velocities at the fluid–structure interface is formulated by formula (10)

$$p\left(x,-\frac{1}{2}\right)=\mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2+\kappa^2\Omega^2}}V\left(x,-\frac{1}{2}\right). \tag{A.12}$$

The dependence of all functions on axial coordinate x is sought in the form $\exp(kx)$:

$$\sigma_x(x)=\sigma_x^0\exp(kx), p_s(x)=P_s\exp(kx), u_0(x)=U_0\exp(kx), v_0(x)=V_0\exp(kx).$$

Thus, the following system of four linear homogeneous algebraic equations in σ_x^0, P_s, U_0, V_0 is obtained from Eqs. (A.7), (A.9), (A.10) and (A.12):

$$\sigma_x^0=\frac{2}{\nu(1+\nu)}V_0-\frac{1-\nu}{\nu}P_s, \tag{A.13a}$$

$$kU_0=\frac{1-\nu}{\nu}2V_0+\frac{2\nu^2+\nu-1}{\nu}P_s, \tag{A.13b}$$

$$k\sigma_x^0+\rho\omega^2U_0=0, \tag{A.13c}$$

$$P_s=\mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2+\kappa^2\Omega^2}}V_0. \tag{A.13d}$$

Isolation of V_0 in equation (A.13a) and utilising (A.13d) yields

$$V_0=\nu kU_0\left[2(1-\nu)-\mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2+\kappa^2\Omega^2}}(2\nu^2+\nu-1)\right]^{-1}. \tag{A14}$$

Then Eq. (A.13a) acquires the form:

$$\sigma_x^0=V_0\left[\frac{2}{\nu(1+\nu)}-\frac{1-\nu}{\nu}\mathrm{i}\bar{\rho}\frac{\Omega^2}{\sqrt{k^2+\kappa^2\Omega^2}}\right]. \tag{A15}$$

Finally, substitution of (A.15)–(A.13c) gives a dispersion relation (24)

$$k^2\left[\frac{2}{1+\nu}-\mathrm{i}\bar{\rho}\Omega^2\frac{(1-\nu)}{\sqrt{k^2+\kappa^2\Omega^2}}\right]\left[2(1-\nu)-\mathrm{i}\bar{\rho}\Omega^2\frac{(1+\nu)(1-2\nu)}{\sqrt{k^2+\kappa^2\Omega^2}}\right]^{-1}+\Omega^2=0.$$

This approximate dispersion equation is obtained as a result of several heuristic assumptions, rather than as a result of consistent asymptotic analysis. Therefore, its validity must be assessed as it is done in Section 4.

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